1. The *parity* $P$ is the improper Lorentz symmetry that reflects the space but not the time, 
$(x, t) \rightarrow (-x, +t)$. This symmetry acts on Dirac spinor fields according to

$$\hat{\Psi}'(-x, +t) = \pm \gamma^0 \hat{\Psi}(+x, +t)$$

(1)

where the overall $\pm$ sign is the *intrinsic parity* of the fermion species.

(a) Verify that the Dirac equation transforms covariantly under (1) and that the Dirac Lagrangian is invariant (apart from $\mathcal{L}(x, t) \rightarrow \mathcal{L}(-x, t)$).

In the Fock space, eq. (1) becomes

$$\hat{P} \hat{\Psi}(x, t) \hat{P} = \pm \gamma^0 \hat{\Psi}(-x, t)$$

(2)

for some unitary operator $\hat{P}$ that squares to one. Let’s find how this operator acts on the particles and their states.

(b) Check that the plane-wave solutions $u(p, s)$ and $v(p, s)$ from the previous homework — set 8, problem 3 — satisfy $u(-p, s) = +\gamma^0 u(p, s)$ and $v(-p, s) = -\gamma^0 v(p, s)$, then use these relations to show that eq. (2) implies

$$\hat{P} \hat{a}_{p, s} \hat{P} = \pm \hat{a}_{-p, +s}, \quad \hat{P} \hat{a}^\dagger_{p, s} \hat{P} = \pm \hat{a}^\dagger_{-p, +s},$$

$$\hat{P} \hat{b}_{p, s} \hat{P} = \mp \hat{b}_{-p, +s}, \quad \hat{P} \hat{b}^\dagger_{p, s} \hat{P} = \mp \hat{b}^\dagger_{-p, +s},$$

(3)

and hence

$$\hat{P} \ket{F(p, s)} = \pm \ket{F(-p, +s)} \quad \text{and} \quad \hat{P} \ket{\bar{F}(p, s)} = \mp \ket{\bar{F}(-p, +s)}.$$  

(4)

Note that the fermion and the antifermion have opposite intrinsic parities!
Now consider the bilinear products of a Dirac field $\Psi(x)$ and its conjugate $\overline{\Psi}(x)$. Generally, such products have form $\overline{\Psi} \Gamma \Psi$ where $\Gamma$ is one of 16 matrices discussed in the previous homework — set 8, problem 1. Altogether, we have

$$S = \overline{\Psi} \Psi, \quad V^\mu = \overline{\Psi} \gamma^\mu \Psi, \quad T^{\mu\nu} = \frac{i}{2} \overline{\Psi} \gamma^{[\mu} \gamma^{\nu]} \Psi, \quad A^\mu = \overline{\Psi} \gamma^5 \gamma^\mu \Psi, \quad P = \overline{\Psi} i \gamma^5 \Psi. \quad (5)$$

(c) Show that all the bilinears (5) are Hermitian.

Hint: despite the Fermi statistics, $\left( \overline{\Psi}_\alpha \Psi_\beta \right)^\dagger = + \overline{\Psi}_\beta \Psi_\alpha$; use this fact to show that $\left( \overline{\Psi} \Gamma \Psi \right)^\dagger = \overline{\Psi} \Gamma \Psi$.

(d) Show that under continuous Lorentz symmetries, the $S$ and the $P$ transform as scalars, the $V^\mu$ and the $A^\mu$ as vectors, and the $T^{\mu\nu}$ as an antisymmetric tensor.

(e) Find the transformation rules of the bilinears (5) under parity and show that while $S$ is a true scalar and $V$ is a true (polar) vector, $P$ is a pseudoscalar and $A$ is an axial vector.

2. The charge conjugation symmetry $\mathbf{C}$ does not transform the space or the time; instead, it exchanges particles with antiparticles, for example the electrons $e^-$ with the positrons $e^+$,

$$\hat{\mathbf{C}} | e^- (p, s) \rangle = | e^+ (p, s) \rangle, \quad \hat{\mathbf{C}} | e^+ (p, s) \rangle = | e^- (p, s) \rangle . \quad (6)$$

In class I have explained that in the fermionic Fock space $\mathbf{C}$ is realized as a unitary operator $\hat{\mathbf{C}} = \hat{\mathbf{C}}^\dagger = \hat{\mathbf{C}}^{-1}$ which acts on the creation and annihilation operators according to

$$\hat{\mathbf{C}} \hat{a}_{p, s} \hat{\mathbf{C}} = \pm \hat{b}_{p, s}, \quad \hat{\mathbf{C}} \hat{b}_{p, s} \hat{\mathbf{C}} = \pm \hat{a}_{p, s}, \quad \hat{\mathbf{C}} \hat{\mathbf{C}} = \pm \hat{\mathbf{C}} . \quad (7)$$

where the overall $\pm$ sign is the intrinsic C-parity which depends on the fermionic species. I have also showed that eqs. (7) imply that the quantum Dirac fields $\hat{\Psi}(x)$ and $\hat{\overline{\Psi}}(x)$
transform under charge conjugation to

\[ \hat{C} \Psi(x) \hat{C} = \pm \gamma^2 \Psi^*(x) \quad \text{and} \quad \hat{C} \bar{\Psi}(x) \hat{C} = \pm \bar{\Psi}(x) \gamma^2. \] (8)

(a) Show that the classical Dirac Lagrangian is invariant under the charge conjugation up to a total spacetime derivative. Note that in the classical limit the Dirac fields anticommute with each other, \( \Psi^*_\alpha \Psi_\beta = -\Psi^*_\beta \Psi_\alpha \). Also, similar to the hermitian conjugation of quantum fields, the complex conjugation of classical fermionic fields reverses their order, \((F_1 F_2)^* = F_2^* F_1^* = -F_1^* F_2^*\).

Next, consider the charge-conjugation properties of the Dirac bilinears (5). To avoid the operator-ordering problems, take the classical limit where \( \Psi(x) \) and \( \Psi^\dagger(x) \) anticommute with each other, \( \Psi^\dagger_\beta \Psi_\alpha = -\Psi^\dagger_\alpha \Psi_\beta \).

(b) Show that \( \mathcal{C} \) turns \( \bar{\Psi} \Gamma \Psi \) into \( \bar{\Psi} \Gamma^c \Psi \) where \( \Gamma^c = \gamma^0 \gamma^2 \Gamma^\top \gamma^0 \gamma^2 \).

(c) Calculate \( \Gamma^c \) for all 16 independent matrices \( \Gamma \) and find out which Dirac bilinears are \( \mathcal{C} \)-even and which are \( \mathcal{C} \)-odd.

3. Now consider a neutral bound state of a charged Dirac fermion \( F \) and the corresponding antifermion, for example a \( q \bar{q} \) meson or a positronium “atom” (a hydrogen-atom-like bound state of \( e^- \) and \( e^+ \)). In the Fock space, such a bound state can be constructed as

\[ |B(p_{\text{tot}} = 0)\rangle = \int \frac{d^3 p_{\text{red}}}{(2\pi)^3} \sum_{s_1, s_2} \psi(p_{\text{red}}, s_1, s_2) \times \hat{a}^\dagger(p_{\text{red}}, s_1) \hat{b}^\dagger(-p_{\text{red}}, s_2) |0\rangle \] (9)

for some wave-function \( \psi \) of the reduced momentum and the two spins.

(a) Suppose this bound state has a definite orbital angular momentum \( L \) and definite net spin \( S \). Show that the intrinsic C-parity and the P-parity of this bound state are

\[ C = (-1)^{L+S}, \quad P = (-1)^{L+1}. \] (10)

(b) Use eqs. (10) to explain why the annihilation rate of the ground 1S state of the positronium “atom” depends on the net spin: the \( S = 0 \) state decays much faster than the \( S = 1 \) state. Note: since the EM fields couple linearly to the electric charges and currents (which are reversed by \( \hat{C} \)), each photon has \( C = -1 \).
4. In the last homework — set 8, problem 2 — we saw that a left-handed Weyl spinor $\psi_L$ is equivalent to the complex conjugate of a right-handed Weyl spinor $\psi_R$ and vice versa. Consequently, a Dirac spinor field $\Psi(x)$ together with its conjugate $\overline{\Psi}(x)$ are equivalent to two left-handed Weyl spinor fields $\chi(x)$ and $\tilde{\chi}(x)$ together with their right-handed conjugates $\sigma_2 \chi^*(x)$ and $\sigma^2 \tilde{\chi}^*(x)$. In the Weyl basis (where $\gamma^5$ is diagonal)

$$\Psi(x) = \begin{pmatrix} \chi(x) \\ -\sigma_2 \tilde{\chi}^*(x) \end{pmatrix}, \quad \overline{\Psi}(x) = \begin{pmatrix} -\tilde{\chi}^T(x) \sigma_2, \chi^T(x) \end{pmatrix}. \quad (11)$$

(a) Show that up to a total derivative,

$$\mathcal{L}_{\text{Dirac}} \equiv \overline{\Psi}(i \partial - m) \Psi = i \chi^\dagger \sigma^\mu \partial_\mu \chi + i \tilde{\chi}^\dagger \sigma^\mu \partial_\mu \tilde{\chi} + m \chi^\dagger \sigma_2 \tilde{\chi} + m \chi^\dagger \sigma_2 \tilde{\chi}^*. \quad (12)$$

Hint: $\sigma_2 \sigma^\mu \sigma_2 = (\sigma^\mu)^* = (\sigma^\mu)^\top$.

Note the $\chi \leftrightarrow \tilde{\chi}$ symmetry of the Lagrangian (12): In the last two terms, the $\sigma^2$ matrix is antisymmetric but the fields are fermionic, hence $\chi^\top \sigma_2 \tilde{\chi} = -\tilde{\chi}^\top \sigma_2 \chi = +\tilde{\chi}^\top \sigma_2 \chi$ and likewise $\chi^\dagger \sigma_2 \tilde{\chi}^* = +\tilde{\chi}^\dagger \sigma_2 \chi^*$.

(b) Express the Dirac bi-linears (5) in terms of the Weyl spinors $\chi$ and $\tilde{\chi}$ (and their hermitian conjugates). For simplicity, assume classical anticommuting fermionic fields.

(c) Work out how the parity $P$, the charge conjugation $C$, and the combined $CP$ symmetry act on the Weyl spinor fields $\chi(x)$ and $\tilde{\chi}(x)$.

Now let’s generalize from two Weyl spinor fields comprising a Dirac field $\Psi$ to any number $N$ of left-handed Weyl spinor fields $\chi_j(x)$ with free Lagrangian

$$\mathcal{L} = \sum_j i \chi_j^\dagger \sigma^\mu \partial_\mu \chi_j + \frac{1}{2} \sum_{j,k} M^{jk} \chi_j^\dagger \sigma_2 \chi_k + \frac{1}{2} \sum_{j,k} M^{*} \chi_j \sigma_2 \chi_k^*. \quad (13)$$

The mass matrix $M^{jk}$ here must be symmetric, $M^{jk} = M^{kj}$, but it may be complex rather than real.
(d) Show that the Weyl equations for the $\chi_j$ fields lead to Klein–Gordon equations

$$\partial^2 \chi_i + (M^*M)_i^j \chi_j = 0, \quad (14)$$

which mean that the physical fermion masses$^2$ are eigenvalues of the $M^*M = M^\dagger M$ matrix.

Hints: use $\sigma_2(\sigma^\mu)^\dagger \sigma_2 = \sigma^\mu$ and $\sigma^\mu \bar{\sigma}^\nu + \bar{\sigma}^\nu \sigma^\mu = 2g^{\mu\nu}$.

Now consider the combined $\text{CP}$ symmetry of the Weyl fermions. The simplest realization of this symmetry acts similarly on all the spinors,

$$\text{CP} : \chi_j(x, t) \rightarrow = \pm i \times \sigma_2 \chi_j^*(-x, +t), \quad \text{same } \pm i \forall j. \quad (15)$$

Note that this realization is slightly different from what we had in part (b) — instead of $\pm$ sign for the $\chi$ and the opposite $\mp$ sign for the $\bar{\chi}$, we now have the same overall factor $\pm i$ for all the $\chi_i$.

(e) Show that the free Lagrangian (13) is invariant under this symmetry if and only if the mass matrix $M^{jk}$ is real. If the mass matrix $M^{jk}$ is complex, we can make it real via some unitary transform of fermions into each other, $\chi_i(x) \rightarrow U_i^j \chi_j(x)$. Consequently, the free Weyl fermions always have a CP symmetry, but its action on the original (un-transformed) spinors becomes

$$\text{CP} : \chi_j(x, t) \rightarrow \sum_k C^k_j \sigma_2 \chi_k^*(-x, +t) \quad (16)$$

for some unitary matrix $C$.

(f) Show that the Lagrangian (13) is invariant under (16) provided the mass matrix $M$ and the unitary matrix $C$ are related by $CM^*C^\dagger = -M$.

* For extra challenge, show that such $C$ matrix exists for any complex symmetric mass matrix $M$.

However, for the interacting fermions, changing the basis and hence the CP action from (15) to (16) may spoil the CP symmetry of the interactions. In class, I shall explain how this happens for the weak interactions.