1. First, a reading assignment: §7.2 of the Peskin & Schroeder textbook about the LSZ reduction formula.

2. Next, a simple exercise about the Yukawa theory. For $M_s > 2m_f$, the scalar particle becomes unstable: it decays into a fermion and an antifermion, $S \to f + \bar{f}$.
   
   (a) Calculate the tree-level decay rate $\Gamma(S \to f + \bar{f})$.
   
   (b) In class, we have calculated
   
   $$\Sigma^{1\text{loop}}(p^2) = \frac{12g^2}{16\pi^2} \int_0^1 d\xi \Delta(\xi) \times \left[ \frac{1}{\epsilon} - \gamma_E + \frac{1}{3} + \log \frac{4\mu^2}{\Delta(\xi)} \right]$$  \hspace{1cm} (1)
   
   $$\Delta(\xi) = m_f^2 - \xi(1-\xi)p^2.$$  \hspace{1cm} (2)
   
   Show that for $p^2 > 4m^2_f$, this $\Sigma_{\Phi}(p^2)$ has an imaginary part and calculate it for $p^2 = M^2_s + i\epsilon$.
   
   Note: at this level, you may neglect the difference between $m_f^{\text{bare}}$ and $m_f^{\text{physical}}$.
   
   (c) Verify that
   
   $$\text{Im} \Sigma^{1\text{loop}}(p^2 = M^2_s + i\epsilon) = -M_s\Gamma_{\text{tree}}(S \to f + \bar{f})$$  \hspace{1cm} (3)
   
   and explain this relation in terms of the optical theorem.

3. Finally, a harder exercise about the scalar $\lambda\phi^4$ theory. As discussed in class, in this theory field strength renormalization begins at two-loop level. Specifically, the 1PI diagram

   $$\text{Diagram}$$  \hspace{1cm} (4)
   
   provides the leading contribution to the $d\Sigma(p^2)/dp^2$ and hence to the $Z - 1$. Your task is to evaluate this contribution.
(a) First, write the two-loop $\Sigma(p^2)$ as an integral over two independent loop momenta, say $q_1^\mu$ and $q_2^\mu$, then use Feynman’s parameter trick — cf. eq. (F.d) of the homework set — to write the product of three propagators as

$$\int\int\int d\xi \, d\eta \, d\zeta \, \delta(\xi + \eta + \zeta - 1) \frac{2}{(D)^3}$$

where $D$ is a quadratic polynomial of the momenta $q_1, q_2, p$. Finally, change the independent momentum variables from $q_1$ and $q_2$ to $k_1 = q_1 + \text{something} \times q_2 + \text{something} \times p$ and $k_2 = q_2 + \text{something} \times p$ to give $D$ a simpler form

$$D = \alpha \times k_1^2 + \beta \times k_2^2 + \gamma \times p^2 - m^2 + i0$$

for some $(\xi, \eta, \zeta)$–dependent coefficients $\alpha, \beta, \gamma$, for example

$$\alpha = (\xi + \zeta), \quad \beta = \frac{\xi \eta + \xi \zeta + \eta \zeta}{\xi + \zeta}, \quad \gamma = \frac{\xi \eta \zeta}{\xi \eta + \xi \zeta + \eta \zeta}.\quad (7)$$

Make sure the momentum shift has unit Jacobian $\partial(q_1, q_2)/\partial(k_1, k_2) = 1$.

Warning: Do not set $p^2 = m^2$ at this stage.

(b) Express the derivative $d\Sigma(p^2)/dp^2$ in terms of

$$\int\int d^4k_1 \, d^4k_2 \frac{1}{D^4}.$$  

Note that although this momentum integral diverges as $k_{1,2} \to \infty$, the divergence is logarithmic rather than quadratic.

(c) To evaluate the momentum integral (8), Wick-rotate the momenta $k_1$ and $k_2$ to the Euclidean space, and then use the dimensional regularization. Here are some useful formulæ for this calculation:

$$\frac{6}{A^4} = \int_0^\infty dt \, t^3 \, e^{-At},\quad (9)$$

$$\int \frac{d^Dk}{(2\pi)^D} e^{-ctk^2} = (4\pi ct)^{-D/2},\quad (10)$$

$$\Gamma(2\epsilon)X^\epsilon = \frac{1}{2\epsilon} - \gamma_E + \frac{1}{2} \log X + O(\epsilon).\quad (11)$$
(d) Assemble your results as
\[
\frac{d\Sigma(p^2)}{dp^2} = -\frac{\lambda^2}{12(4\pi)^4} \int\int\int_{\xi,\eta,\zeta \geq 0} d\xi d\eta d\zeta \frac{\delta(\xi + \eta + \zeta - 1) \times \frac{\xi \eta \zeta}{(\xi \eta + \xi \zeta + \eta \zeta)^3} \times 
\times \left( \frac{1}{\epsilon} - 2\gamma_E + 2 \log \frac{4\pi \mu^2}{m^2} + \log \frac{(\xi \eta + \xi \zeta + \eta \zeta)^3}{(\xi \eta + \xi \zeta + \eta \zeta - \xi \eta \zeta (p^2/m^2))^2} \right)}{\xi \eta \zeta (\xi \eta + \xi \zeta + \eta \zeta)^3 (\xi \eta + \xi \zeta + \eta \zeta - \xi \eta \zeta (p^2/m^2))^2}.
\]

(e) Before you evaluate the Feynman parameter integral (12) — which looks like a frightful mess — make sure it does not introduce its own divergences. That is, without actually calculating the integrals
\[
\int\int\int_{\xi,\eta,\zeta \geq 0} d\xi d\eta d\zeta \frac{\delta(\xi + \eta + \zeta - 1) \times \frac{\xi \eta \zeta}{(\xi \eta + \xi \zeta + \eta \zeta)^3} \times 
\times \log \frac{(\xi \eta + \xi \zeta + \eta \zeta)^3}{(\xi \eta + \xi \zeta + \eta \zeta - \xi \eta \zeta (p^2/m^2))^2} \right)}{\xi \eta \zeta (\xi \eta + \xi \zeta + \eta \zeta)^3 (\xi \eta + \xi \zeta + \eta \zeta - \xi \eta \zeta (p^2/m^2))^2}.
\]

make sure that they converge. Pay attentions to the boundaries of the parameter space and especially to the corners where \(\xi, \eta \to 0\) while \(\zeta \to 1\) (or \(\xi, \zeta \to 0\), or \(\eta, \zeta \to 0\)).

This calculation shows that
\[
\frac{d\Sigma}{dp^2} = \text{constant} \times \frac{1}{\epsilon} + a_{\text{finite function}}(p^2)
\]
and hence
\[
\Sigma(p^2) = (\text{a divergent constant}) + (\text{another divergent constant}) \times p^2 + a_{\text{finite function}}(p^2)
\]
up to the two-loop order. In fact, this behavior persists to all loops, so all the divergences of \(\Sigma(p^2)\) may be canceled with just two counterterms, \(\delta^m\) and \(\delta^Z \times p^2\).

* Optional exercise: Evaluate the integrals (13) for \(p^2 = m^2\) and show that
\[
\int\int\int_{\xi,\eta,\zeta \geq 0} d\xi d\eta d\zeta \frac{\delta(\xi + \eta + \zeta - 1) \times \frac{\xi \eta \zeta}{(\xi \eta + \xi \zeta + \eta \zeta)^3} \times 
\times \log \frac{(\xi \eta + \xi \zeta + \eta \zeta)^3}{(\xi \eta + \xi \zeta + \eta \zeta - \xi \eta \zeta (p^2/m^2))^2} \right)}{\xi \eta \zeta (\xi \eta + \xi \zeta + \eta \zeta)^3 (\xi \eta + \xi \zeta + \eta \zeta - \xi \eta \zeta (p^2/m^2))^2}.
\]

Do not try to do this calculation by hand — it would take way too much time. Instead,
use *Mathematica* or equivalent software. To help it along, replace the \((\xi, \eta, \zeta)\) variables with \((x, w)\) according to

\[
\begin{align*}
\xi &= w \times x, & \eta &= w \times (1 - x), & \zeta &= 1 - w,
\end{align*}
\]

\[
\int \int \int d\xi d\eta d\zeta \delta(\xi + \eta + \zeta - 1) = \int_0^1 dx \int_0^1 dw \, w,
\]

(17)

then integrate over \(w\) first and over \(x\) second.

Alternatively, you may evaluate the integrals like this numerically. In this case, don’t bother changing variables, just use a simple 2D grid spanning a triangle defined by \(\xi + \eta + \zeta = 1, \xi, \eta, \zeta \geq 0\); modern computers can sum up a billion grid points in less than a minute. But watch out for singularities at the corners of the triangle.

(f) Finally, calculate the field strength renormalization factor

\[
Z = \left[1 - \frac{d\Sigma}{dp^2}\right]^{-1}
\]

(18)
to the two-loop order. Use the bare perturbation theory, *i.e.* divergent \(\lambda_{\text{bare}}\) and \(m^2_{\text{bare}}\) instead of the counterterms.

Note: the derivative \(d\Sigma/dp^2\) in eq. (18) should be evaluated at \(p^2 = M^2_{\text{ph}}\) — the physical mass\(^2\) of the scalar particle, but to the leading approximation we may let \(M^2_{\text{ph}} \approx m^2\) and set \(p^2 = m^2\) in eq. (12). This simplifies the second integral (13) a little bit — *cf.* eqs. (16) — although it’s still a royal pain to calculate.