1. Consider the Wess–Zumino model, a QFT comprising a Majorana spinor $\Psi(x)$, a real scalar $\Phi_1(x)$, and a real pseudoscalar $\Phi_2(x)$, all massless. The physical Lagrangian

$$L = \frac{i}{2} \bar{\Psi} \not{\partial} \Psi + \frac{1}{2} (\partial_\mu \Phi_1)^2 + \frac{1}{2} (\partial_\mu \Phi_2)^2 - \frac{g}{2} \bar{\Psi} (\Phi_1 + i\gamma^5 \Phi_2) \Psi - \frac{\lambda}{8} (\Phi_1^2 + \Phi_2^2)^2, \quad (1)$$

has a global $U(1)$ chiral symmetry, which acts as

$$\Psi \rightarrow \exp(i\theta \gamma^5) \Psi, \quad (\Phi_1 + i\Phi_2) \rightarrow \exp(-2i\theta \gamma^5) \times (\Phi_1 + i\Phi_2). \quad (2)$$

Wess and Zumino found that for $\lambda = g^2$, the renormalization of this theory is particularly simple, but at first they did not know why. Salam and Strathdee realized there must be a symmetry behind this simplicity, and after working very hard to find it, they discovered the supersymmetry.

Thanks to the chiral symmetry, the WZ model needs only 5 independent counterterms, namely $\delta^\theta, \delta^\lambda, \delta^Z_\phi, \delta^Z_\psi$, and $\delta^m_\phi$, but no $\delta^m_\psi$! In general, $\delta^m_\phi = O(\Lambda^2)$ while the other counterterms are $O(\log(\Lambda/E))$.

(a) For $\lambda = g^2$, the quadratic divergence of the two-scalar 1PI amplitude vanishes. Instead, $\Sigma^{\text{loops}}_{\phi}(p^2) = p^2 \times O(\log \Lambda^2/p^2)$ and hence $\delta^m_\phi = 0$ while $\delta^Z_\phi = O(\log \Lambda/E)$. Show that this is true at the one-loop level.

Note: Feynman rules for the Majorana fermions are similar to those for the Dirac fermions (same propagators, vertices, and external leg factors), but there is an extra factor $\frac{1}{2}$ for each closed fermionic loop. \textit{i.e.}, $-\frac{1}{2} \text{tr}(\cdots)$ instead of $-\text{tr}(\cdots)$.

(b) Next, calculate the infinite parts of the other 4 counterterms at the one-loop level. Proceed similarly to homework #17, and do not hesitate to recycle similar calculations instead of redoing them from scratch. Do not assume $\lambda = g^2$ at this stage.

(c) Calculate the anomalous dimensions of the scalar and fermionic fields to order $O(g^2, \lambda)$ and show that $\gamma_\phi = \gamma_\psi$. Note: at the one-loop level this is true for any $\lambda$, but at the higher loop levels $\gamma_\phi = \gamma_\psi$ only when $\lambda = g^2$. 


(d) Calculate the beta-functions \( \beta_g(g, \lambda) \) and \( \beta_\lambda(g, \lambda) \) to one-loop order for general \( \lambda \) and \( g \). Then show that

\[
\text{for } \lambda = g^2, \quad \beta_\lambda(\lambda = g^2) = 2g \times \beta_g(\lambda = g^2). \tag{3}
\]

Note: because of supersymmetry, this relation holds true to all orders of the perturbation theory. But in this exercise, you should check it at the one-loop level only.

(e) Show that the relation (3) implies that if \( \lambda(E_0) = g^2(E_0) \) for any particular energy \( E_0 \), then \( \lambda(E) = g^2(E) \) for all energies \( E \).

(f) Finally, consider the renormalization group flow in the \((g^2, \lambda)\) plane. In the UV \( \rightarrow \) IR direction, is the \( \lambda = g^2 \) line attractive or repulsive?

2. And now a reading assignment: *Quantum Mechanics and Path Integrals* by Feynman & Hibbs. Read all you can about care and use of Path Integrals. After the break, I will talk about “path” integrals in QFT, and it would help if you already know something about path integrals in the ordinary QM.