1. First, a bit of group theory. Consider a generic simple non-abelian compact Lie group $G$ and its generators $T^a$. For a suitable normalization of the generators, 

$$\text{tr}_{(r)}(T^aT^b) \equiv \text{tr}\left(T^a_{(r)}T^b_{(r)}\right) = R(r)\delta^{ab}$$ (1)

where the trace is taken over any complete multiplet $(r)$ — irreducible or reducible, it does not matter — and $T^a_{(r)}$ is the matrix representing the generator $T^a$ in that multiplet. The coefficient $R(r)$ in eq. (1) depends on the multiplet $(r)$ but it’s the same for all generators $T^a$ and $T^b$. The $R(r)$ is called the index of the multiplet $(r)$.

The (quadratic) Casimir operator $C_2 = \sum_a T^a T^a$ commutes with all the generators, $\forall b, \ [C_2, T^b] = 0$. Consequently, when we restrict this operator to any irreducible multiplet $(r)$ of the group $G$ it becomes a unit matrix times some number $C(r)$. In other words,

$$\text{for an irreducible } (r), \ \sum_a T^a_{(r)}T^a_{(r)} = C(r) \times 1_{(r)}.\quad (2)$$

For example, for the isospin group $SU(2)$, the Casimir operator is $C_2 = \vec{I}^2$, the irreducible multiplets have definite isospin $I = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and $C(I) = I(I+1)$.

(a) Show that for any irreducible multiplet $(r)$,

$$\frac{R(r)}{C(r)} = \frac{\dim(r)}{\dim(G)}.\quad (3)$$

In particular, for the $SU(2)$ group, this formula gives $R(I) = \frac{1}{3}I(I+1)(2I+1)$.

(b) Suppose the first three generators of $G$ generate an $SU(2)$ subgroup. Show that if a multiplet $(r)$ of $G$ decomposes into several $SU(2)$ multiplets of isospins $I_1, I_2, \ldots, I_n$, then

$$R(r) = \sum_{i=1}^n \frac{1}{3}I_i(I_i + 1)(2I_i + 1).\quad (4)$$

(c) Now consider the $SU(N)$ group with an obvious $SU(2)$ subgroup of matrices acting

only on the first two components of a complex $N$-vector. This complex $N$-vector is
called the fundamental multiplet (of the \(SU(N)\)) and denoted \(N\) or \(\mathbf{N}\). As far as the \(SU(2)\) subgroup is concerned, \((N)\) comprises one doublet and \(N - 2\) singlets, hence

\[
R(N) = \frac{1}{2} \quad \text{and} \quad C(N) = \frac{N^2 - 1}{2N}.
\]  
(5)

Show that the adjoint multiplet of the \(SU(N)\) decomposes into one \(SU(2)\) triplet, \(2(N - 2)\) doublets, and \((N - 2)^2\) singlets, therefore

\[
R(\text{adj}) = C(\text{adj}) \equiv C(G) = N.
\]  
(6)

Hint: \((N) \times (\bar{N}) = (\text{adj}) + (1)\).

(d) The symmetric and the anti-symmetric 2–index tensors form irreducible multiplets of the \(SU(N)\) group. Find out the decomposition of these multiplets under the \(SU(2) \subset SU(N)\) and calculate their respective indices \(R\) and Casimirs \(C\).

2. Now let’s apply this group theory to physics. Consider quark-antiquark pair production in QCD, specifically \(u \bar{u} \rightarrow d \bar{d}\). There is only one tree diagram contributing to this process,

![Diagram](attachment:image.png)

(7)

Evaluate this diagram, then sum/average the \(|\mathcal{M}|^2\) over both spins and colors of the final/initial particles to calculate the total cross section. For simplicity, you may neglect the quark masses.

Note that the diagram (7) looks exactly like the QED pair production process \(e^- e^+ \rightarrow \text{virtual } \gamma \rightarrow \mu^- \mu^+\), so you can re-use the QED formula for summing/averaging over the spins. But in QCD, you should also sum/average over colors of all the quarks, and that’s the whole point of this exercise.
3. In problem 2 from previous homework set you should have calculated the annihilation of a scalar ‘quark’ and an ‘antiquark’ into a pair of gluons. To convert the annihilation amplitude into a cross-section we need to sum/average over the colors of all the particles. As a first step in this direction, it’s convenient to write the amplitude as

\[ \mathcal{M}(j + i \rightarrow a + b) = F \times \{T^a, T^b\}_j^i + iG \times [T^a, T^b]_j^i \]  

(8)

where \( F \) and \( G \) are some functions of momenta and polarizations of the vector particles while \( a, b, i, \) and \( j \) are the color indices of the four particles. Specifically, the \( a \) and \( b \) colors of the gauge bosons run over the adjoint multiplet of \( G \), the \( j \) index of the scalar ‘quark’ runs over the multiplet \((r)\), and the \( i \) index of the scalar ‘antiquark’ runs over the conjugate multiplet \((\bar{r})\).

(a) Show that the annihilation amplitude indeed has form (8) and write down the coefficients \( F \) and \( G \) as explicit functions of the particles momenta and polarizations.

(b) Next, let us sum the \( |\mathcal{M}|^2 \) over the gauge boson’s colors and average over the scalars’ colors. Show that

\[
\frac{1}{\dim^2(r)} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{C(r)}{\dim(r)} \times (4C(r) \times |F|^2 + C(\text{adj}) \times (|G|^2 - |F|^2)) .
\]

(9)

In particular, for scalars in the fundamental representation of the \( SU(N) \) gauge group,

\[
\frac{1}{N^2} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{N^2 - 1}{2N^2} \left( \frac{N^2 - 2}{N} |F|^2 + N|G|^2 \right) .
\]

(10)

(c) Evaluate \( F \) and \( G \) in the center of mass frame, where the vector particles’ polarizations \( e_{1,2}^\mu = (0, e_{1,2}) \) are purely spatial and transverse to the vectors’ momenta \( \pm \mathbf{k} \). For simplicity, use planar rather than circular polarizations.

(d) Assemble your results and calculate the (polarized, partial) cross-section for the annihilation process.
4. In class, I calculated the (infinite parts of the) $\delta_2$ and $\delta_1$ counterterms for the quarks. Your task is to calculate the analogous $\delta_2^{(gh)}$ and $\delta_1^{(gh)}$ counterterms for the ghosts fields.

(a) Draw one-loop diagrams whose divergences are cancelled by the $\delta_2^{(gh)}$ and $\delta_1^{(gh)}$, and calculate the group factors for each diagrams.

(b) Calculate the momentum integrals for the diagrams. Focus on the UV divergences and ignore the finite parts of the integrals.

(c) Assemble your results and show that the difference $\delta_1^{(gh)} - \delta_2^{(gh)}$ for the ghosts is exactly the same as the $\delta_1 - \delta_2$ difference for the quarks.