QED Feynman Rules in the Counterterm Perturbation Theory

The simplest version of QED (Quantum ElectroDynamics) has only 2 field types — the EM field $A^\mu$ and the electron field $\Psi$ — and the physical Lagrangian

$$\mathcal{L}_{\text{phys}} = -\frac{1}{4} F_{\mu\nu}^2 + \overline{\Psi} (i \gamma^\mu D_\mu - m_e) \Psi = -\frac{1}{4} F_{\mu\nu}^2 + \overline{\Psi} (i \not{\partial} - m) \Psi + e A_\mu \overline{\Psi} \gamma^\mu \Psi. \quad (1)$$

Renormalizing the fields according to $A^\mu_{\text{bare}} = \sqrt{Z_3} \times A^\mu$, $\Psi_{\text{bare}} = \sqrt{Z_2} \times \Psi$ and substituting the fields into the bare Lagrangian, we obtain

$$\mathcal{L}_{\text{bare}} = -\frac{Z_3}{4} \times F_{\mu\nu}^2 + i Z_2 \times \overline{\Psi} \not{\partial} \Psi - Z_2 m_{\text{bare}} \times \overline{\Psi}$$

$$+ \left( Z_1 e \overset{\text{def}}{=} Z_2 \sqrt{Z_3} \right) \times A_\mu \overline{\Psi} \gamma^\mu \Psi$$

$$= \mathcal{L}_{\text{phys}} + \mathcal{L}_{\text{counterterms}} \quad (2)$$

where $\mathcal{L}_{\text{phys}}$ is precisely as in eq. (1) while the counterterms comprise

$$\mathcal{L}_{\text{counterterms}} = -\frac{1}{4} \delta_3 \times F_{\mu\nu}^2 + i \delta_2 \times \overline{\Psi} \not{\partial} \Psi - \delta_m \times \overline{\Psi} \Psi + e \delta_1 \times A_\mu \overline{\Psi} \gamma^\mu \Psi \quad (3)$$

for

$$\delta_3 = Z_3 - 1, \quad \delta_2 = Z_2 - 1, \quad \delta_1 = Z_1 - 1, \quad \delta_m = Z_2 m_{\text{bare}} - m_{\text{phys}}. \quad (4)$$

In the counterterm perturbation theory, we take the free Lagrangian to be

$$\mathcal{L}_{\text{free}} = -\frac{1}{4} F_{\mu\nu}^2 + \overline{\Psi} (i \not{\partial} - m) \Psi \quad (5)$$

(where $m$ is the physical mass of the electron) while all the other terms in the bare Lagrangian — the physical coupling $e A_\mu \overline{\Psi} \gamma^\mu \Psi$ and all the counterterms (3) — are treated as perturbations. Consequently, the QED Feynman rules have the following propagators and vertices:

- The electron propagator

$$\begin{array}{c}
\alpha \\
\overrightarrow{p}
\end{array} \begin{array}{c}
\beta \\
\end{array} = \left[ \frac{i}{\not{p} - m + i0} \right]_{\alpha\beta} = \frac{i(p + m)_{\alpha\beta}}{p^2 - m^2 + i0} \quad (6)
\end{array}$$

where $\alpha$ and $\beta$ are the Dirac indices, usually not written down.
• The photon propagator

\[ \mu \nu \kappa = \frac{-i g^{\mu \nu}}{k^2 + i0} \quad (7) \]

in the Feynman gauge. In a more general Lorentz-invariant gauges, the photon propagator becomes

\[ \mu \nu \kappa = \frac{-i k^\mu}{k^2 + i0} \times \left(g^{\mu \nu} + (\xi - 1) \frac{k^\mu k^\nu}{k^2 + i0}\right) \quad (8) \]

for some gauge-dependent parameter \( \xi \).

• The physical vertex

\[ \begin{array}{c}
\alpha \\
\downarrow \\
\beta \\
\downarrow \\
\mu \\
\end{array} = (+ie\gamma^\mu)_{\alpha\beta} . \quad (9) \]

The Dirac indices \( \alpha \) and \( \beta \) of the fermionic lines are usually not written down.

★ And then there are three kinds of the counterterm vertices:

\[ \begin{array}{c}
\alpha \\
\downarrow \\
\beta \\
\downarrow \\
\mu \\
\end{array} = +ie\delta_1 \times (\gamma^\mu)_{\alpha\beta} , \quad (10) \]

\[ \begin{array}{c}
\alpha \\
\downarrow \\
\beta \\
\downarrow \\
\end{array} = +i(\delta_2 \times \not{n} - \delta_m)_{\alpha\beta} , \quad (11) \]

\[ \begin{array}{c}
\mu \\
\downarrow \\
\nu \\
\end{array} = -i\delta_3 \times (g^{\mu \nu} k^2 - k^\mu k^\nu) . \quad (12) \]